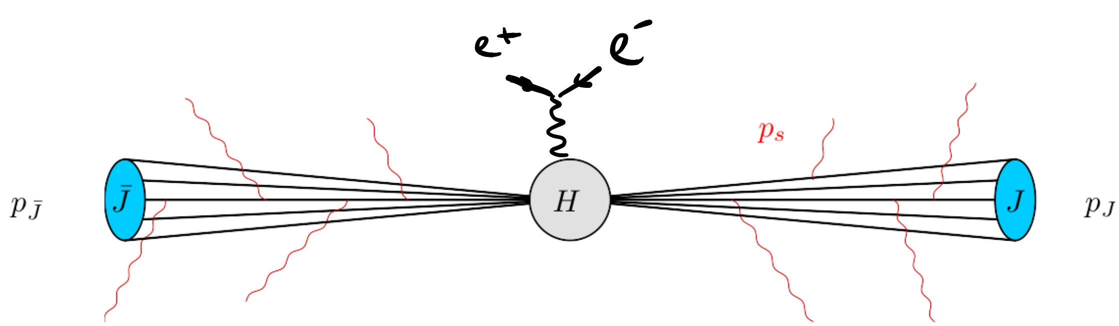
Soft-Collinear Effective Theory

(see 1803.04310, 1410.1892)

Last week, we analyzed soft-photon radiation in QED and derived a factorization theorem

$$\sigma = H(Q) \cdot S(E_s)$$

in massive QED. In QCD (and massless QED), often also the collinear region is relevant. Factorization then takes the schematic form



$$\sigma = H(Q^2)\, J(p_J^2)\, J(p_{\bar{J}}^2)\, S(\Lambda_s^2)$$

with $\Lambda_s^2 \sim p_J^2\, p_{\bar{J}}^2 / Q^2$ (see later)

An important technical tool we encountered last week is the method of regions: one can obtain the expansion of loop integrals by

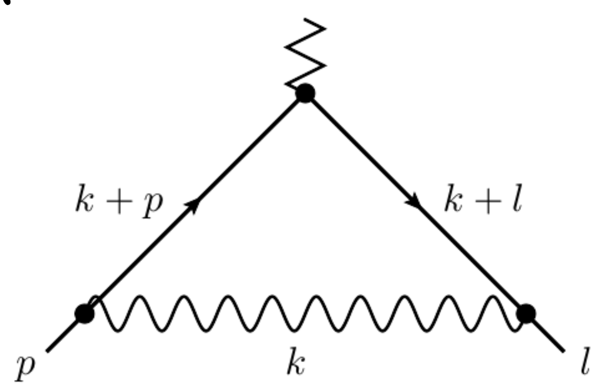
a.) Expanding the integrand in different momentum regions

b.) Integrating over the full momentum space $\int d^d k$

c.) Adding up the different pieces

Sudakov form factor : Method of Region

We now apply this technique to the simplest example for which collinear physics plays a role, the Sudakov form factor. Based on our results, we then construct an effective theory which implements the expansion.



$$P^2 = -p^2 \; ; \; L^2 = -l^2 \; ; \; Q^2 = -(p-l)^2$$

Consider expansion $P^2 \sim L^2 \ll Q^2$

For the moment, we only consider the associated scalar integral, which reads

$$I = \int d^d k \frac{1}{k^2 (k+p)^2 (k+l)^2}$$

For the kinematics under consideration p & l have large energies & small virtuality

To expand around the limit, it is useful to introduce light-like reference vectors

$$n^\mu = (1, 0, 0, 1) \approx p^\mu / p^0$$

$$\bar{n}^\mu = (1, 0, 0, -1) \approx l^\mu / l^0$$

An arbitrary momentum can be decomposed as

$$q^\mu = n\cdot q \frac{\bar{n}^\mu}{2} + \bar{n}\cdot q \frac{n^\mu}{2} + q_\perp^\mu$$

two orthogonal directions

$n\cdot q_\perp = \bar{n} q_\perp = 0$

$$= q_+^\mu + q_-^\mu + q_\perp^\mu$$

Note that

$$q^2 = q_+\cdot q_- + q_\perp^2$$

Let us introduce an expansion parameter

$$\lambda^2 \sim p^2/Q^2 \sim L^2/Q^2$$

(λ is just a book-keeping device)

Then $p^2 = p_+ p_- + p_\perp^2 \sim \lambda^2 Q^2$

while $(p-\ell)^2 \approx -2\,p_-\cdot \ell_+ \approx Q^2$

The components of p^μ and ℓ^μ therefore scale as:

$$q^\mu \sim (n\cdot q \,,\, \bar{n}\cdot q \,,\, q_\perp)$$

$$p^\mu \sim (\lambda^2 \,,\, 1 \,,\, \lambda)\, Q$$

$$\ell^\mu \sim (1 \,,\, \lambda^2 \,,\, \lambda)\, Q$$

Let us now consider different scalings of the loop momentum:

	$n\cdot k$	$\bar{n}\cdot k$	$k_\perp$	
hard (h):	1	1	1	Q
collinear to p^μ (c):	λ^2	1	λ	Q
" ℓ^μ ($\bar{c}$):	1	λ^2	λ	Q
soft (s):	λ^2	λ^2	λ^2	Q

Expanding the loop integrand in each region and performing the integrations all of these regions contribute, while all other scalings $k^\mu \sim (\lambda^a, \lambda^b, \lambda^c)Q$ give scaleless integrals upon expanding.

(exercise: pick a scaling and check!)

Let us perform the expansion of the integrand in the different regions.

At leading power

$$k \text{ hard}: \quad (k+p)^2 = (k+p_-)^2 + O(\lambda)$$

$$(k+\ell)^2 = (k+\ell_+)^2 + O(\lambda)$$

$$k \text{ collinear to } p: \quad (k+p)^2 = (k+p)^2 \quad \text{(no expansion!)}$$

$$(k+\ell) = 2k_- \cdot \ell_+ + O(\lambda)$$

k soft : $(k+p)^2 = 2p_-\cdot k_+ + p^2 + O(\lambda^3)$

$(k+\ell)^2 = 2\ell_+ k_- + \ell^2 + O(\lambda^3)$

The expanded loop integrals are

$$I_h = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{(k^2+i0)\,(k^2+2k_-\cdot l_+ +i0)\,(k^2+2k_+\cdot p_- +i0)}$$

$$I_h = \frac{\Gamma(1+\varepsilon)}{2l_+\cdot p_-}\frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{\mu^2}{\underbrace{2l_+\cdot p_-}_{\approx Q^2}}\right)^{\varepsilon} \sim (Q^2)^{-\varepsilon}$$

hard scale

$$I_c = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{(k^2+i0)\,(2k_-\cdot l_+ +i0)\,[(k+p)^2+i0]}$$

$$I_c = -\frac{\Gamma(1+\varepsilon)}{2l_+\cdot p_-}\frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{\mu^2}{P^2}\right)^{\varepsilon} \sim (P^2)^{-\varepsilon}$$

collinear scale

$$I_s = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{(k^2+i0)\,(2k_-\cdot l_+ +l^2+i0)\,(2k_+\cdot p_- +p^2+i0)}$$
$$= -\frac{\Gamma(1+\varepsilon)}{2l_+\cdot p_-}\Gamma(\varepsilon)\Gamma(-\varepsilon)\left(\frac{2l_+\cdot p_-\mu^2}{L^2P^2}\right)^{\varepsilon}. \sim (\Lambda_s^2)^{-\varepsilon}$$

Note: soft scale

$$\Lambda_s^2 = \frac{L^2P^2}{Q^2} \ll L^2 \sim P^2$$

After the expansion, these are all single-scale integrals. Note that all of them involve divergences, while the original integral is finite. Let's expand in ε and add up

$$
\begin{aligned}
I_h &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}+\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}-\frac{\pi^2}{6}\right)\\
I_c &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{2}\ln^2\frac{\mu^2}{P^2}+\frac{\pi^2}{6}\right)\\
I_{\bar{c}} &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{L^2}-\frac{1}{2}\ln^2\frac{\mu^2}{L^2}+\frac{\pi^2}{6}\right)\\
I_s &= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2Q^2}{L^2P^2}+\frac{1}{2}\ln^2\frac{\mu^2Q^2}{L^2P^2}+\frac{\pi^2}{6}\right)
\end{aligned}
$$

$$
I_{\mathrm{tot}} = \frac{1}{Q^2}\left(\ln\frac{Q^2}{L^2}\ln\frac{Q^2}{P^2}+\frac{\pi^2}{3}\right).
$$

The sum is finite and agrees with the expansion of the original integral!

The cancellation is quite nontrivial since the $1/\varepsilon \ln(\dots)$ divergences involve different scales.

Effective Lagrangian

We now construct an effective theory along the same lines as in the previous lecture: the expanded diagrams are viewed as effective theory diagrams. We introduce fields for the different low-energy regions and construct $\mathcal{L}_{eff}$ whose Feynman rules give the diagrams in the different regions. At tree level, we can substitute

$$\begin{aligned} \psi &\rightarrow \psi_c + \psi_{\bar{c}} + \psi_s \\ A^\mu &\rightarrow A^\mu_c + A^\mu_{\bar{c}} + A^\mu_s \end{aligned} \qquad (*)$$

in the QCD Lagrangian and expand in λ.

To do so, we need to know how the different components of the fields scale.

For the gluon field

$$\langle 0 | T\{ A^a_\mu(x) A^b_\nu(0) \} | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i\delta^{ab}}{k^2} \left\{ -g^{\mu\nu} + \xi \frac{k^\mu k^\nu}{k^2} \right\} \times e^{-ikx}$$

So typically $A_\mu \sim k^\mu$, i.e.

$$(n\cdot A_s,\ \bar{n} A_s,\ A_s^\perp) \sim (\lambda^2, \lambda^2, \lambda^2)$$

$$(n\cdot A_c,\ \bar{n} A_c,\ A_c^\perp) \sim (\lambda^2, 1, \lambda)$$

For the soft fermion field

$$\langle 0 | T\{ \psi_s(x) \bar{\psi}_s(0) \} | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i\not{k}}{k^2} e^{-ikx}$$

$$\sim \lambda^8 \frac{\lambda^2}{\lambda^4} = \lambda^6$$

$$\rightarrow \psi_s(x) \sim \lambda^3$$

For collinear fermions, the situation is more complicated:

$$\not{k} = k\cdot n \frac{\not{\bar{n}}}{2} + k\cdot\bar{n}\frac{\not{n}}{2} + \not{k}_\perp$$

$$\lambda^0 \qquad \lambda^2 \qquad \lambda$$

To separate the different contributions we split the field

$$\psi_c = \xi_c + \eta_c = P_+\psi_c + P_-\psi_c$$

with $P_+ = \frac{\not{n}\not{\bar{n}}}{4}$; $P_- = \frac{\not{\bar{n}}\not{n}}{4}$.

The fulfill $P_+ + P_- = 1$; $P_\pm^2 = P_\pm$.

Then

$$\langle 0|T\{\xi_c(x)\bar{\xi}_c(0)\}|0\rangle = \int\frac{d^4k}{(2\pi)^4} e^{-ikx} \underbrace{\frac{\not{n}\not{\bar{n}}}{4}\frac{i\not{k}}{k^2}\frac{\not{\bar{n}}\not{n}}{4}}_{i\bar{n}\cdot p\frac{\not{n}}{2}\frac{1}{p^2}}$$

$$\sim\lambda^4 \qquad \cdot\frac{1}{\lambda^2} \sim \lambda^2$$

$\Rightarrow \xi_c \sim \lambda$. Similarly $\eta_c \sim \lambda^2$.

Now that we know how the fields scale, we plug (*) into the QCD action and get

$$\underset{\text{purely soft}}{\underset{\uparrow}{S}} = S_s + S_c + S_{\bar{c}} + \underset{\text{s-c interactions.}}{\underset{\uparrow}{S_{s+c}}} + S_{s+\bar{c}} + \dots$$

The purely sof part

$$S_s = \underset{\sim\lambda^{-8}}{\underset{\uparrow}{\int d^4x}} \, \overbrace{\bar{\psi}_s \, i\underset{\nwarrow \; i\partial_\mu + gA_s}{\not{D}_s} \, \psi_s}^{(\lambda^2)^4} - \frac{1}{4} G^a_{s\,\mu\nu} G^{\mu\nu a}_s$$

has exactly the same form as the usual QCD Lagrangian. All terms are $O(\lambda^0)$

Also the collinear part is a copy of QCD, but we should plug in the decomposition of the fermion field

$$\mathcal{L}_c = (\bar{\xi}_c + \bar{\eta}_c)\left[i n\cdot D_c \frac{\bar{\not{n}}}{2} + i\bar{n}\cdot D \frac{\not{n}}{2} + i\not{D}_\perp \right](\xi_c + \eta_c)$$
$$- \frac{1}{4} G^a_{c\mu\nu} G_c^{\mu\nu a} .$$

$$= \bar{\xi}_c\, in\cdot D \frac{\bar{\not{n}}}{2} \xi_c + \bar{\eta}_c\, i\bar{n}\cdot D \frac{\not{n}}{2} \eta_c + \bar{\eta}_c i\not{D}_{c\perp} \xi_c + \bar{\xi}_c i\not{D}_\perp \eta_c$$
$$- \frac{1}{4} G^a_{c\mu\nu} G_c^{\mu\nu a} .$$

This form is inconvenient: $\xi_c \sim \lambda$, $\eta_c \sim \lambda^2$ and the two fields mix. To solve this, one shifts

$$\eta_c \longrightarrow \eta_c - \frac{\bar{\not{n}}}{2} \frac{1}{i\bar{n}\cdot D_c} i\not{D}_{c\perp} \xi$$

large momentum $\sim Q$

to complete the square. This yields

$$\mathcal{L}_c = \bar{\xi}_c \frac{\bar{\not{n}}}{2}\left[in\cdot D_c + i\not{D}_{c\perp} \frac{1}{i\bar{n}\cdot D_c} i\not{D}_{c\perp} \right] \xi_c$$

$$+\,\bar{\eta}_c \frac{\bar{n}\!\!\!/}{2}\, i\bar{n}\cdot D\, \eta_c \;-\; \frac{1}{4} G^a_{c\,\mu\nu} G_c^{\mu\nu a}$$

Then one integrates out the η_c-field. This leaves a determinant $\det\left(\frac{\bar{n}\!\!\!/}{2} i\bar{n}\cdot D\right)$. This determinant is trivial. To see this note that it is gauge invariant and manifestly trivial in the gauge $\bar{n}\cdot A = 0$.

Next, let's consider S_{s+c}. Getting the leading-power terms is actually quite simple because:

a.) ψ_s is power suppressed, compared to collinear quarks: no ψ_s at leading power.

b.) $\bar{n}\cdot A_s \ll \bar{n}\cdot A_c$, $A_s^\perp \ll A_c^\perp$. Only $n\cdot A_s \sim n\cdot A_c$ arises at leading power.

Taken together, these imply that the s-c interactions can be obtained by substituting

$$A^\mu_c \to A^\mu_c + n\cdot A_s \frac{\bar{n}^\mu}{2}$$

in $\mathcal{L}_c$. The interaction Lagrangian thus takes the form

$$S_{c+s} = \int d^4x \, \bar{\xi}_c(x) \frac{\not{\bar{n}}}{2} n\cdot A_s(x) \, \xi_c(x)$$

+ "gluon terms"

Since this term contains collinear fields and

$$p^\mu_c + p^\mu_s \sim p^\mu_c \sim (\lambda^2, 1, \lambda)$$

$$\rightsquigarrow \qquad x^\mu \sim (1, 1/\lambda^2, 1/\lambda)$$

We can thus perform a derivative expansion, which is the analogue of expanding in the soft momentum, e.g.

$$x_\perp \cdot \partial_\perp \phi_s \sim x^\perp \cdot p_s^\perp \phi_s$$

$$\frac{1}{\lambda} \quad \lambda^2 \quad \sim \lambda$$

$$x_+ \cdot \partial_- \phi_s \sim x_+ \cdot p_{s-} \phi_s$$

$$1 \cdot \quad \lambda^2 \quad \sim \lambda$$

Hence:

$$S_{c+s} = \int d^4x\, \bar{\xi}_c(x) \frac{\not{\bar{n}}}{2} \overbrace{\Big(1 + \underset{\lambda}{x_\perp \cdot \partial_\perp} + \underset{\lambda^2}{x_+ \partial_-} + \underset{\cdots}{\cdots}\Big)}^{\text{Taylor series}}$$

$$\cdot\, n \cdot A_s(x)\Big|_{x=x_-} \xi_c(x)$$

$$= \int d^4x\, \bar{\xi}_c(x) \frac{\not{\bar{n}}}{2} n \cdot A_s(x_-)\, \xi_c(x)$$

$\rightsquigarrow$ In s-c interactions, we must replace $x^\mu \to x_-^\mu$ at leading power.

"Multipole expansion"

Summary:

also contains c+s through n.D.

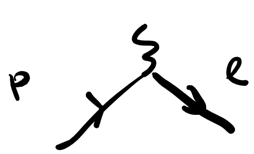
$$\mathcal{L}_{\text{SCET}} = \bar{\psi}_s i\not{D}_s \psi_s + \bar{\xi}_c \frac{\not{\bar{n}}}{2}\left[in\cdot D + i\not{D}_{c\perp}\frac{1}{i\bar{n}\cdot D_c} i\not{D}_{c\perp}\right]\xi_c - \frac{1}{4}\left(F^{s,a}_{\mu\nu}\right)^2 - \frac{1}{4}\left(F^{c,a}_{\mu\nu}\right)^2$$

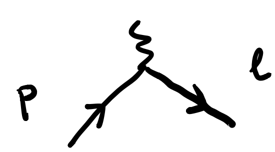
$$+ \mathcal{L}_{\bar{c}} + \mathcal{L}_{\bar{c}+s}$$ ← same form as $\mathcal{L}_c + \mathcal{L}_{c+s}$ but with $n \leftrightarrow \bar{n}$

where $$in\cdot D = in\cdot\partial + g\,n\cdot A_c(x) + g\,n\cdot A_s(x_-)$$

Vector current in SCET

The final piece of the effective theory is the vector current. At leading order the matching is trivial





$$J^\mu = \bar{\psi}\gamma^\mu\psi \longrightarrow J = \bar{\xi}_{\bar{c}}\gamma^\mu \xi_c + \ldots$$

We can simplify this a bit further

$$\bar{\xi}_{\bar{c}}\, \gamma^\mu\, \xi_c = \bar{\xi}_{\bar{c}} \Big[\underbrace{n^\mu \frac{\bar{n}\!\!\!/}{2}}_{=0} + \underbrace{\bar{n}^\mu \frac{n\!\!\!/}{2}}_{=0} + \gamma_\mu^\perp \Big] \xi_c$$

$$= \bar{\xi}_{\bar{c}}\, \gamma_\mu^\perp\, \xi_c$$

However, to perform the matching properly, we should write the most general leading-power operator. Since the momentum component $\bar{n}\cdot p$ of a collinear field is large

$$\bar{n}\cdot\partial\, \phi_c \sim \lambda^0 Q\, \phi_c$$

we must allow for operators with arbitrarily many derivatives!

An efficient way of doing so is to use the identities

$$\phi_c(x + t\bar{n}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\bar{n}\cdot\partial)^n \phi_c(x)$$

and

$$\int dt\, C(t)\, \phi_c(x + t\bar{n}) = \sum_{n=0}^{\infty} \frac{C_n}{n!} (\bar{n}\cdot\partial)^n \phi_c(x)$$

where $C_n = \int dt\, C(t)\, t^n$.

Instead of introducing infinitely many Wilson coefficients C_n, we smear the field over the light-cone with a function $C(t)$.

In a gauge theory we must make sure that we maintain gauge invariance when smearing the operator, e.g. when writing

$$\bar{\xi}_c(x+t\bar{n})\,U(x+t\bar{n},\,t)\,\xi_c(x)$$

(which is the matrix element defining parton distribution functions!) we need to have a link field U which connects the two fields at different points. We can use the Wilson line

path ordering

matrix field $A^a_\mu \cdot t^a$

$$U_c(x+t\bar{n},x) = \mathcal{P}\exp\left[ig\int_0^t dt'\,\bar{n}\cdot A_c(x+t'\bar{n})\right]$$

to achieve this. Under a collinear gauge transformation $V_c(x) = \exp(i\alpha_c^a(x)t^a)$

$$U_c(x+t\bar{n}, x) \longrightarrow V(x+t\bar{n})\, U_c(\dots)\, V^\dagger(x).$$

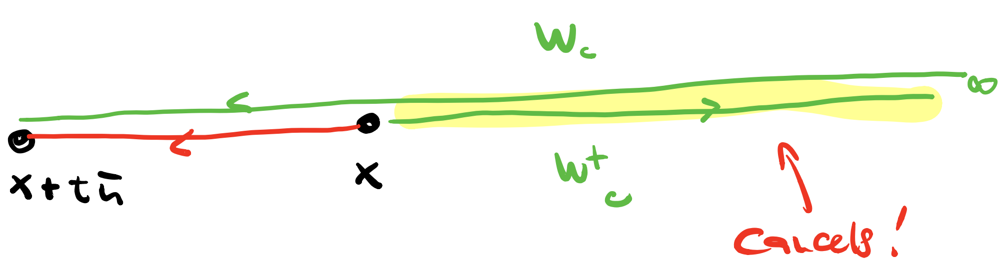
In SCET, one rewrites the link $U_c(x+tn, x)$ in terms of the object

$$W_c(x) = U(x, x-\infty\bar{n})$$

in terms of which we can write

$$U(x+t\bar{n}, x) = W_c(x+t\bar{n})\, W_c^\dagger(x)$$

Pictorially:



Using W_c, we can define the building blocks

$$\chi_c(x) = W_c^{\dagger}(x)\,\xi_c(x)$$

$$\mathcal{A}^{\mu}_c(x) = W_c^{\dagger}\left(D_c^{\mu} W_c\right)$$

which are invariant under gauge transformations which vanish at infinity.

After this long preparation, we are finally ready to write down the leading-power SCET operator:

Wilson coefficient

$$J^{\mu}(0) = \int ds \int dt\; C_V(s,t)\,\bar{\chi}_{\bar{c}}(s n)\,\gamma_{\perp}^{\mu}\,\chi_c(t\bar{n})$$

At tree-level $C_V(s,t) = \delta(s)\,\delta(t)$. To understand the meaning of C_V, let us take a matrix element

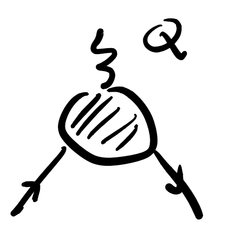
$$\langle q(\ell) | J^\mu(0) | q(p) \rangle = \int ds \int dt \; C_V(s,t)$$

$$\cdot \; \bar{u}(\ell) \gamma_\perp^\mu u(p) \, e^{-isn\ell} \, e^{it\bar{n}\cdot p}$$

$$= \tilde{C}_V(\underbrace{n\ell \, \bar{n}p}_{Q^2}) \, \bar{u}(\ell) \gamma_\perp^\mu u(p)$$

Note that SCET is invariant under the transformation $n \to \alpha n$, $\bar{n} \to 1/\alpha \, \bar{n}$. Because of this the coefficient $\tilde{C}_V$ only depends on $Q^2 = n\ell \; \bar{n}\cdot p$. To determine $\tilde{C}_V$ at one computes

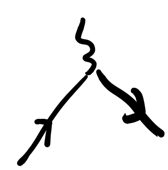
QCD



$p^2 = 0$ $\quad$ $\ell^2 = 0$

$\equiv$

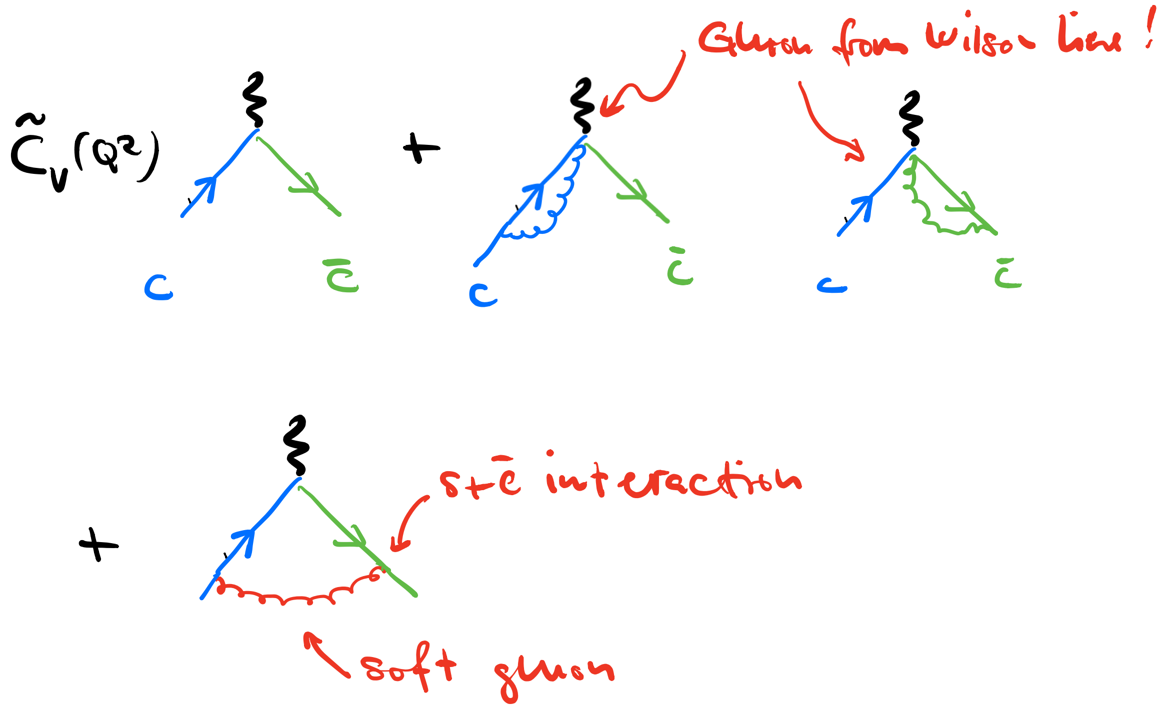
SCET

$\tilde{C}_V(Q^2)$ 

$+ \, 0$ ← collinear & soft loops are scaleless!

The on-shell form factor directly corresponds to $\tilde{C}_V$!

It is now an interesting exercise to verify that all regions occurring in the diagrammatic analysis using the method of regions are fully reproduced by SCET. This is indeed the case, see 1410.1892



Decoupling transformation & factorization

Next, we'll proceed as in the QED case:

we'll perform a field redefinition

multipole expansion!

$$\xi_c(x) = S_n(x_-)\, \xi_c^{(0)}(x)$$

$$A_c^\mu = S_n(x_-)\, A_c^{(0)\mu}\, S_n^\dagger(x_-)$$

with the soft Wilson line

$$S_n(x) := \mathbb{P} \exp\left[ig \int_{-\infty}^{0} ds\, n\cdot A_s(x+sn) \right]$$

matrix $A_\mu^a \cdot t^a$

path ordering for the matrices

The result is the same as for the QED case: the soft field decouples from the leading-power $\mathcal{L}_{c+s}$:

$$\bar{\xi}_c \, i n \cdot D \frac{\not{\bar{n}}}{2} \xi_c = \bar{\xi}_c^{(0)} \, i n \cdot D_c^{(0)} \xi_c^{(0)}$$

Also, again as in the QED example, the soft Wilson lines appear in the operator :

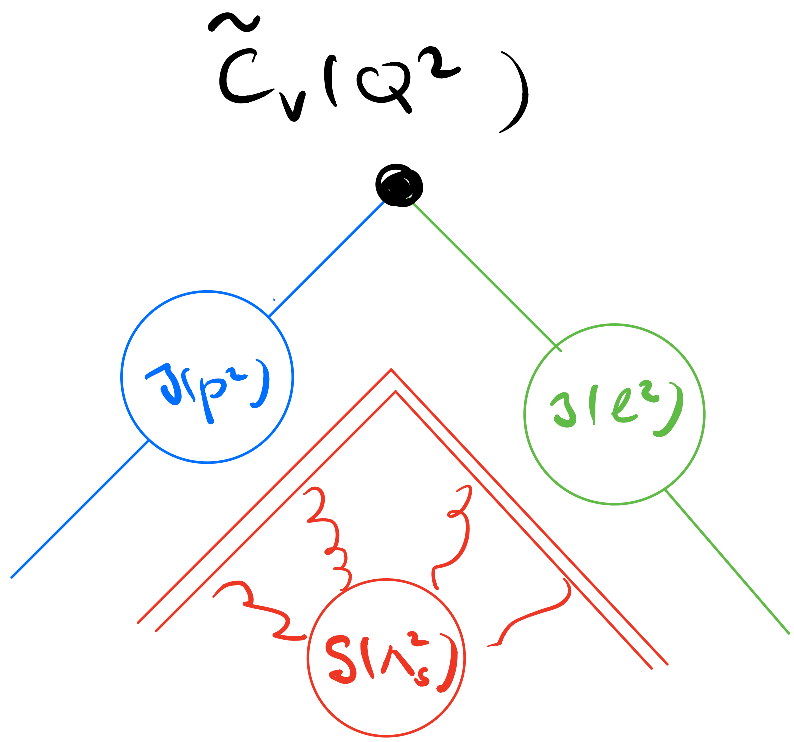
$$J^\mu(0) = \int ds \int dt \, C_V(s,t)$$

$$\underbrace{\bar{\chi}_{\bar{c}}^{(0)}(sn)} \; \underbrace{S_{\bar{n}}^{\dagger}(0) \, \gamma_\perp^\mu \, S_n(0)} \; \underbrace{\chi_c^{(0)}(t\bar{n})}$$

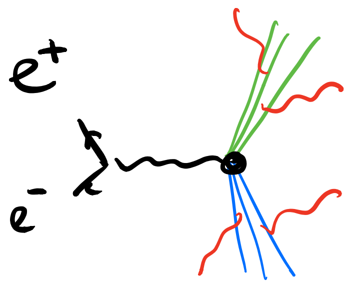
$$x_- = \bar{n} \cdot x \frac{n^\mu}{2}$$

After the decoupling, the 3 different fields no longer interact: We have factorized the Sudakov form factor.

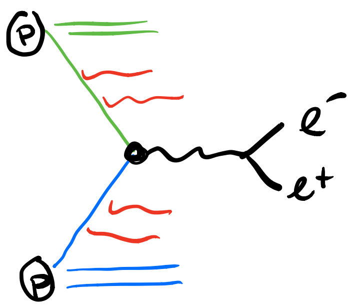
Schematically:



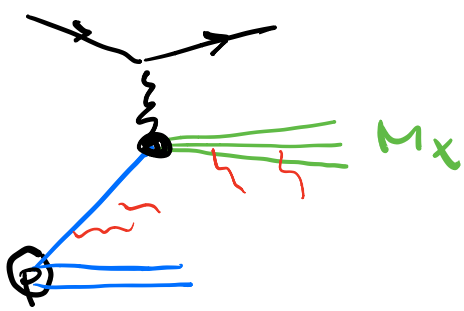
Unfortunately we don't have time to discuss applications of SCET beyond the Sudakov form factor, however, the analysis of the vector form factor forms the basis of many applications, e.g.



Event-shapes in 2-jet production (e.g. thrust)



Drell-Yan process ($pp \to e^+e^- + X$) near threshold.



Deep Inelastic Scattering ($e^-p \to e^- + X$) for small M_X